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2003 J. Phys. A: Math. Gen. 36 L255

(http://iopscience.iop.org/0305-4470/36/17/101)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) L255–L261 PII: S0305-4470(03)59549-X

LETTER TO THE EDITOR

Brownian motion in wedges, last passage time and the second arc-sine law

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Received 12 February 2003, in final form 19 March 2003
Published 16 April 2003
Online at stacks.iop.org/JPhysA/36/L255

Abstract

We consider a planar Brownian motion starting from O at time $t = 0$ and stopped at $t = 1$ and a set $F = \{OI_i; i = 1, 2, \ldots, n\}$ of n semi-infinite straight lines emanating from O. Denoting by g the last time when F is reached by the Brownian motion, we compute the probability law of g. In particular, we show that, for a symmetric F and even n values, this law can be expressed as a sum of arcsin or $(\arcsin)^2$ functions. The original result of Levy is recovered as the special case $n = 2$. A relation with the problem of reaction–diffusion of a set of three particles in one dimension is discussed.

PACS numbers: 02.70.−c, 03.65.−w

The first arc-sine law gives the distribution of the number of positive partial sums in a sequence of independent and identically distributed random variables. It was first discovered by Levy in his study of the linear Brownian motion and then discussed a lot for its relevance to the coin-tossing game [1] and also in the wider context of occupation time distributions [2]. The second arc-sine law, also discovered by Levy [3], provides information on the last passage time which can be stated as follows. Consider a linear Brownian motion $B(\tau)$ starting at 0 at time $t = 0$ and stopped at time t and let g be the last time when 0 is visited. The random variable

$$g = \sup\{\tau < t, B(\tau) = 0\} \tag{1}$$

satisfies[3]

$$P(g < u) = \frac{2}{\pi} \arcsin \sqrt{\frac{u}{t}} \tag{2}$$

with the density

$$\mathcal{P}(u) = \frac{1}{\pi} \frac{1}{\sqrt{u(t-u)}}. \tag{3}$$

[3] The fact that the occupation time $\int_0^t d\tau\, \theta[B(\tau)]$ and the last passage time g are identically distributed is a very striking result which has been discussed in the mathematical literature under the name of 'fluctuations identities' [4].

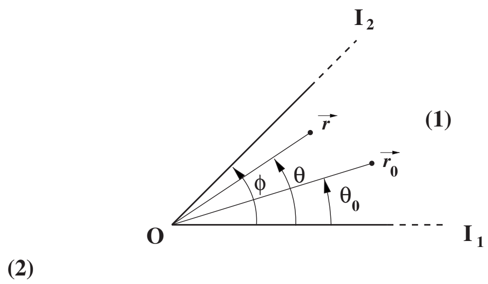

Figure 1. The frontier $F = \{OI_1, OI_2\}$ of the wedge divides the plane into two regions, **(1)** and **(2)**.

Over the years, this result has been extended in several different directions (see, for instance, [5, 6]) and is still a subject of active research in probability [7]. Generalizations of the first arc-sine law have also been considered in different contexts (one-dimensional diffusion in a random medium [8], Brownian motion on graphs [9] and, also, in two dimensions [10]).

The purpose of this letter is to present a two-dimensional generalization of the law (2). As a by-product of this result we also derive an explicit expression for the first passage time distribution which is relevant for a problem of reaction–diffusion involving three identical particles.

Exit problems for Brownian motion have a rich history and several applications in physics (see, for instance, [11]). They are in particular related to problems of capture of independent Brownian particles diffusing on the line. This connection was first anticipated by Arratia [12] and then discussed in the mathematics [13, 14] and physics literature [11, 15–17] mainly in the context of reaction–diffusion models. In the case of three particles, the process $(x_1(t) - x_2(t), x_2(t) - x_3(t))$ defines a certain diffusion in a quadrant of R^2. By a suitable transformation, this process can be mapped on a diffusion inside a wedge whose angle depends on the diffusion constants. Using this correspondence, it has been shown that the first passage time through the wedge gives the survival probability; a quantity which decays with a power law which only depends on the angle of the wedge [16, 18]. At the end of this work, we exploit this correspondance to compute exactly (and not only asymptotically) the first collision time distribution for a three-particle problem. Our approach is based on an identity relating the first passage and last passage distributions which has an interesting probabilistic interpretation [19].

To begin with, let us start by considering, as in figure 1, a wedge of apex O and angle ϕ with a boundary $F = \{OI_i; i = 1, 2\}$ and a two-dimensional Brownian motion $\vec{r}(t)$ starting from O at $t = 0$ and stopped at t somewhere in the plane. We denote by g the last time when F is visited and compute the probability $P(g < u)$. Due to the scaling property of the Brownian motion, this distribution is a function of the reduced variable u/t. In the following, we will for simplicity set $t = 1$.

Suppose that the particle reaches some point $\vec{r}_0$ at time $t = u$ (see figure 1). Clearly, if $\vec{r}_0$ belongs to region **(1)** (resp. **(2)**), the particle must stay in **(1)** (resp. **(2)**) between $t = u$ and $t = 1$ in order to satisfy the condition $g < u$. We can therefore write

$$P(g < u) = P^{(1)}(u) + P^{(2)}(u). \tag{4}$$

Expressing the fact that the propagation is free between $t = 0$ and $t = u$ and that the particle has not hit the boundary between $t = u$ and $t = 1$, we get

$$P^{(i)}(u) = \int_{(i)} \mathrm{d}^2\vec{r}_0 \int_{(i)} \mathrm{d}^2\vec{r}\, \frac{1}{2\pi u}\, \mathrm{e}^{-\frac{r_0^2}{2u}}\, G^{(i)}(\vec{r}, 1; \vec{r}_0, u) \qquad i = 1, 2. \tag{5}$$

The propagator $G^{(1)}$ satisfying the diffusion equation with Dirichlet boundary conditions on F is given by

$$G^{(1)} = \frac{2}{\phi(1-u)} \sum_{m=1}^{\infty} \sin \frac{m\theta\pi}{\phi} \sin \frac{m\theta_0\pi}{\phi} \, \mathrm{e}^{-\frac{r^2+r_0^2}{2(1-u)}} I_{\frac{m\pi}{\phi}} \left(\frac{r r_0}{1-u} \right) \tag{6}$$

where I_ν is a modified Bessel function and the notation are defined in figure 1.

Performing the spatial integrations in (5), we get [20]

$$P^{(1)}(u) = \frac{1}{\pi\phi} \sum_{p,k=0}^{\infty} u^{p\frac{\pi}{\phi}+k+\frac{\pi}{2\phi}} \frac{\left[\Gamma\left(p\frac{\pi}{\phi}+k+\frac{\pi}{2\phi}\right)\right]^2}{\Gamma\left(\frac{2p\pi}{\phi}+\frac{\pi}{\phi}+k+1\right)} \frac{1}{k!} \tag{7}$$

$P^{(2)}$ is obtained by the change $\phi \to 2\pi - \phi$ in (7). Therefore, for arbitrary values of ϕ the law $P(g < u)$ is written in terms of a double series.

As a check, let us first consider the special case $\phi = \pi$. We may write

$$P(g < u) = 2P^{(1)}(u) = \frac{2}{\pi^2} u^{1/2} \sum_{p,k=0}^{\infty} \frac{u^{p+k}}{k!} \frac{[\Gamma(p+k+1/2)]^2}{(2p+k+1)!} \tag{8}$$

$$= \frac{2}{\pi} \arcsin \sqrt{u}. \tag{9}$$

The fact that one recovers Levy's second arc-sine law is not surprising since, when $\phi = \pi$, F divides the plane into two half-planes. Therefore, the component of the Brownian motion parallel to F factorizes and plays no role: we are thus left with a one-dimensional problem.

Coming back to general values of ϕ, we can derive the behaviour of the probability density $\mathcal{P}\left(\equiv \frac{\mathrm{d}P(g<u)}{\mathrm{d}u}\right)$ when $u \to 0$ and $u \to 1$. By using (4) and (7), one gets a power-law behaviour when $u \to 0^+$

$$\mathcal{P}(u) \sim \frac{1}{\pi} u^{-1/2} \qquad \text{for} \quad \phi = \pi \tag{10}$$

$$\mathcal{P}(u) \sim C(\mu) u^{\frac{\mu}{2}-1} \qquad \text{for} \quad \phi \neq \pi \tag{11}$$

with

$$C(\mu) = \frac{\mu^2}{2\pi^2} \frac{\left[\Gamma\left(\frac{\mu}{2}\right)\right]^2}{\Gamma(\mu+1)} \tag{12}$$

and

$$\mu = \frac{\pi}{2\pi - \phi} \qquad \text{when} \quad 0 < \phi < \pi \tag{13}$$

$$\mu = \frac{\pi}{\phi} \qquad \text{when} \quad \pi < \phi < 2\pi. \tag{14}$$

Now, for the limit $u \to 1^-$, using asymptotic expansions for Γ functions and also an equivalence between series and integrals, we get

$$\mathcal{P}(u) \sim \frac{1}{\pi} \frac{1}{\sqrt{1-u}} \tag{15}$$

i.e. the same behaviour as for (3). Expression (15) does not depend on ϕ and we have already seen that $\phi = \pi$ gives Levy's law. Remark that $u \to 1^-$ corresponds to Brownian curves that stop close to F. Therefore, between $t = u$ and $t = 1$, the Brownian particle only 'sees' an

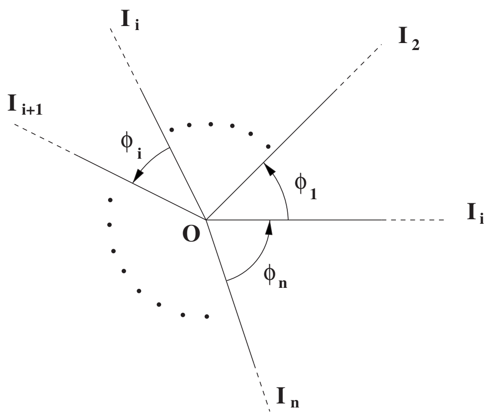

Figure 2. F consists of n semi-infinite straight lines starting from O.

infinitesimal part of F, i.e. a straight line as for $\phi = \pi$. This is, in our opinion, why the result (15) does not depend on ϕ. Actually, it only depends on the fact that the plane is divided by F into two regions. We will come back to this point later on.

To go further, let us remark that for $F = \{OI_i;\, i = 1, 2, \ldots, n\}$ as in figure 2, (4) becomes simply

$$P(g < u) = \sum_{i=1}^{n} P^{(i)}(u). \tag{16}$$

(Replace ϕ by ϕ_i in (7) in order to get $P^{(i)}$.)

Let us now specialize to the situation of figure 3 when F is symmetric and n is even ($n \equiv 2l$). In that case, F consists of l infinite straight lines crossing at point O and dividing the plane into $2l$ equal angular sectors, each one of angle $\phi = \pi/l$.

Equation (16) is written as

$$P(g < u) \equiv P_l(u) = \frac{2}{\pi^2} l^2 u^{l/2} \sum_{p=0}^{\infty} u^{lp} \sum_{k=0}^{\infty} \frac{[\Gamma(lp + k + l/2)]^2}{(2lp + l + k)!} \frac{u^k}{k!} \tag{17}$$

l being an integer, we can sum the series and, finally, get

$$P_l(u) = \frac{2l}{\pi} \left(\sum_{k=0}^{l-1} \arcsin\left(\sqrt{u} \cos \frac{2\pi k}{l} \right) \right) \qquad l \text{ odd} \tag{18}$$

$$P_l(u) = \frac{2l}{\pi^2} \left(\sum_{k=0}^{l-1} (-1)^k \left(\arcsin\left(\sqrt{u} \cos \frac{\pi k}{l} \right) \right)^2 \right) \qquad l \text{ even} \tag{19}$$

which is the central result of this paper.

We remark that the correct small u behaviour for $P_l(u)$ follows from the two identities

$$\sum_{k=0}^{l-1} \left(\cos \frac{2\pi k}{l} \right)^m = 0 \qquad l \text{ odd} \quad m = 1, 3, \ldots, l-2 \tag{20}$$

$$\sum_{k=0}^{l-1} (-1)^k \left(\cos \frac{\pi k}{l} \right)^m = 0 \qquad l \text{ even} \quad m = 0, 2, 4, \ldots, l-2. \tag{21}$$

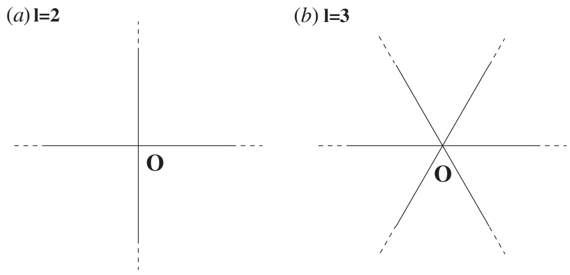

Figure 3. F is symmetric. The analytic form of $P(g < u)$ will depend on the parity of l. Thus, it will be different for cases (a) and (b). For further explanations, see the text.

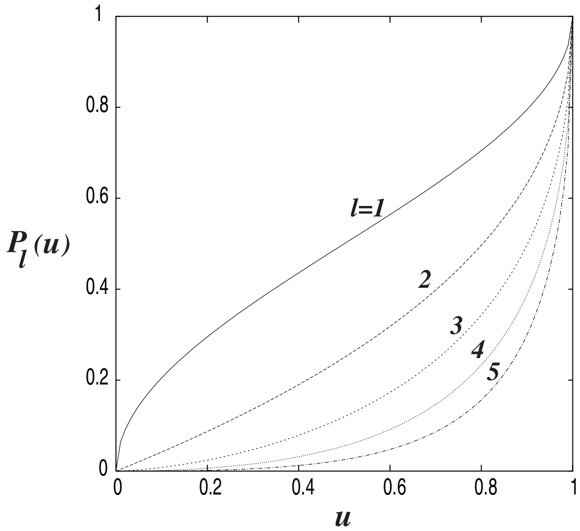

Figure 4. The distribution functions $P_l(u)$ for $l = 1, \ldots, 5$.

In particular

$$P_1(u) = \frac{2}{\pi} \arcsin \sqrt{u} \tag{22}$$

$$P_2(u) = \frac{4}{\pi^2} (\arcsin \sqrt{u})^2 = P_1^2 \tag{23}$$

$$P_3(u) = \frac{6}{\pi} \left(\arcsin \sqrt{u} - 2 \arcsin \frac{\sqrt{u}}{2} \right) \tag{24}$$

$$P_4(u) = \frac{8}{\pi^2} \left((\arcsin \sqrt{u})^2 - 2 \left(\arcsin \sqrt{\frac{u}{2}} \right)^2 \right) \tag{25}$$

$$P_5(u) = \frac{10}{\pi} \left(\arcsin \sqrt{u} - 2 \arcsin \left(\cos \frac{\pi}{5} \sqrt{u} \right) + 2 \arcsin \left(\cos \frac{2\pi}{5} \sqrt{u} \right) \right). \tag{26}$$

These functions are displayed in figure 4.

As expected, Levy's second arc-sine law is recovered in (22). Moreover, the result (23) is straightforward since, when $l = 2$, the two components of the Brownian motion factorize.

Thus, for $l = 2$, $(\arcsin)^2$ functions appear. What is surprising is that they will appear each time l is even while being absent when l is odd.

For the probability density, $\mathcal{P}_l\ \left(\equiv\frac{dP_l}{du}\right)$, with (18) and (19), we obtain

$$\mathcal{P}_l(u) \sim \frac{l}{\pi}\frac{1}{\sqrt{1-u}} \qquad \text{when} \quad u \to 1^-. \tag{27}$$

This is consistent with (15) that corresponds to $l = 1$.

We now present a formula which relates the first passage and the last passage time distributions. The starting point is (4) and (5) which may be rewritten as

$$P(g < u) = \int \Pr(T > 1 - u|r_0)\frac{1}{u}\,e^{-\frac{r_0^2}{2u}}\, r_0\, dr_0 \tag{28}$$

where $\Pr(T > (1 - u)|r_0)$ is the probability distribution of the first passage time T through F, given that the process starts at r_0. Then, by scaling one has

$$\Pr(T > (1-u)|r_0) = \Pr\left(T > \frac{(1-u)}{{r_0}^2}|1\right). \tag{29}$$

By a simple change of variables it follows that

$$P\left(g < \frac{1}{1+t}\right) = \int \Pr\left(T > \frac{t}{2x}|1\right) e^{-x}\, dx. \tag{30}$$

Therefore

$$P\left(g < \frac{1}{1+t}\right) = E\left(e^{-\frac{t}{2T}}\right) \tag{31}$$

which is a relation between the first passage characteristic function for a process starting in the wedge at a distance $r_0 = 1$ from the apex O and the probability distribution of the last passage time. Interestingly enough, this formula can also be derived in a more intrinsic fashion using only time inversion and scaling [19]. As an application, let us derive the density of first passage time in a wedge of angle $\phi = \frac{\pi}{3}$. In the context of the capture problem mentioned in the introduction, this corresponds to a set of three identical and independent particles [13]. In this case, the distribution $P(g)$ is given in equation (24). By an inverse Laplace transform (31) gives the density of first passage time:

$$f(T) = \frac{6}{\pi^{\frac{3}{2}} T}\, e^{-\frac{1}{2T}} \left(\int_0^{\sqrt{\frac{1}{2T}}} e^{y^2}\, dy - 2\int_0^{\sqrt{\frac{1}{8T}}} e^{y^2}\, dy\right). \tag{32}$$

One can check that this formula is in agreement with (16) of [13] which expresses the first collision time probability for a given set of initial conditions. By averaging this formula over the angle and setting $r = 1$ one recovers (32).

Acknowledgment

We are particularly grateful to Professor Marc Yor for introducing us to the subject and for pointing out (31).

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